A Puzzle about Points

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> ...that for a discrete manifold the principle of measurement is already contained in the concept of this manifold, but that for a continuous one it must come from elsewhere. Either therefore the reality which underlies space must form a discrete manifold, or we must seek the ground of its metric relations elsewhere...
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> "On The Hypotheses Which Lie at the Bases of Geometry"
> Bernhard Riemann

1 Problems with Points, Old and New

There is a strong case to be made that if there are any physical objects or regions of space, then they are made of dimensionless, partless, zero-size points. But there are problems with points, or, more exactly, with the assumption that they are what physical objects or regions of space are made of. Difficulties emerge when considering the fact that if there are any physical objects or regions of space, such things have sizes, some of which are positive.[1] I think at least one of these difficulties is real, that it presents a genuine and serious puzzle. But it is not one that has received any serious attention.

Zeno contended that nothing of positive size is composed of points. One of his arguments was rather simple, and was later stated pithily and emphatically by Pierre Bayle [1710, 3077]: "For the most inapprehensive Capacities may apprehend with the utmost certainty, if they consider it with a little attention that several nullities of Extension joined together will never make an Extension." We might put it less pithily and more carefully as follows: for any collection of things, if the (arithmetical) sum of their sizes is x, then the size of their (mereological or set-theoretic) sum is at most x. But for any collection of points no matter how large, the (arithmetical) sum of their sizes is 0, since every point (or its singleton set) is of size 0. So for any collection of points, the size of their (mereological or set-theoretic) sum is at most 0.

The conclusion of the argument, if true, would indeed be problematic for points. But its second premise is false. There are some collections of points whose sizes *have* no arithmetical sum. In particular, an uncountable collection of numbers has no well-defined sum, so the sizes of uncountably many points have no well-defined sum. And physical objects or regions of space, if they are of positive size,

[1] Alongside such size-related problems, there might be other, topological problems: Grunbaum [1952] addresses the problem of how a region of positive dimensions can be the set-theoretic sum of zero-dimensional subsets; and Zimmerman [1996] argues that if extended physical objects were made of points, then they would be precluded from coming into contact with one another just because of the shapes of their surfaces or the structure of their outermost parts, which Zimmerman thinks impossible. Cf. Sider [2000].

are all made of uncountably many points. The size of a physical object or region of space is thus not determined by the size and number of its ultimate parts.[2]

More recently, Peter Forrest [2004] has suggested that a specific result from the mathematical theory of measure creates trouble for points. As Banach and Tarski showed, if measure is finitely additive and the Axiom of Choice is true, then there are regions of $\mathbb{R}^3$ that do not have any size at all. By the same token, if physical space is a three-dimensional manifold of points, and the size of such regions is finitely additive, and the Axiom of Choice is true, then there are regions of space that have no size. But, Forrest contends, every region of space has some size or other. As he puts it,

> My case against the orthodoxy is a simple one. I start from the Banach-Tarski theorem...We have an interesting theorem if we are considering sets of triples of reals. But if we hold that these sets of coordinate triples represent real regions then the Banach-Tarski theorem becomes the Banach-Tarski paradox. (352)

I'm not convinced that every region of space must have some size. Sure, ordinary regions of space better have sizes, but it's not clear to me that *every* region, no matter how cockeyed or diffuse, no matter whether it could ever be occupied by a physical object, imagined by folks like us, or painted purple, must have some size or other. There might be no answer to the question, "How much purple paint do I need to exactly cover that region?" but I didn't expect there to always be.

So much for Zeno and Banach-Tarski. There is, however, a problem with points. If the standard reply to Zeno is right, and the size of a physical object or region of space is not determined by the size and number of its ultimate parts, then what does determine its size? You might reasonably assume that nothing does, that a thing's size is a fundamental feature of that thing. Or, at the very least, that when it comes to a region of space, nothing *external to that region* goes toward determining its size, that its size is entirely determined by how it is in itself.[3] But the standard mathematical theory of measure, together with some plausible metaphysical assumptions, implies that it's not so: facts about the size of a given region are partly about that region's relation to *other* regions. Such facts, or, more exactly, relations, are extrinsic.[4]

[2] See Grunbaum [1952] for a classic statement of this response. Cf. Sherry [1988].

[3] Even if you think that *occupants* of regions have their sizes only in virtue of exactly occupying a region that has that size – in a way analogous to Bradford Skow's [2007] claim about shapes – you are likely inclined to think that the regions they occupy have their sizes intrinsically. Going forward, I will ignore physical objects – except to illustrate my claims – but everything I say about regions made of points and the extrinsicality of their size applies, *a fortiori*, to physical objects made of points.

[4] A word on terminology: I use 'relations' to cover both monadic properties (thought of as 1-place relations) and polyadic relations.

This generate a puzzle about points. If regions of space are made of points, then size relations are extrinsic. If those relations are extrinsic, then their distribution should be settled by the distribution of intrinsic *non*-size relations. But they aren't. Even once the sub-measure-theoretic structure of a physical space is settled – including its topological and affine structure – its measure-theoretic structure still needs settling. That's very puzzling. It suggests that nothing made of points could have any size at all.

In the next section I will elaborate the assumptions that constitute our puzzle. In the concluding section I will consider a reply.

2 The Puzzle

2.1 Assumption 1

The first assumption is as follows:

> (Size Extrinsic) All possible pointy size relations are extrinsic

A *size* relation is one that either (a) specifies the relative size(s) of one or more of its relata (a 'narrow size relation', or a 'size' for short), or (b) entails a narrow size relation. Each of these open sentences expresses some such relation: 'x is a one hundred meter long stretch of track,' 'x has twice the area as y,' and 'x is a wooden two-by-four'.[5] A *pointy* (narrow) size relation is a (narrow) size relation that entails that all of its relata are composed of points. And a *possible* pointy (narrow) size relation is a pointy (narrow) size relation that is possibly instantiated.

An extrinsic relation is one that isn't intrinsic, where the rough idea is that a relation is intrinsic if whenever it is instantiated, it is instantiated solely in virtue of how its bearer is. Restricting our talk of relations to so-called qualitative relations – and all our talk of relations both heretofore and henceforth is hereby restricted in that way – we can say that a relation is intrinsic just in the case that necessarily, for anything that instantiates it, necessarily any intrinsic duplicate of that thing instantiates it as well. In short, a relation is intrinsic iff it never differs between possible intrinsic duplicates.[6]

So, Size Extrinsic amounts to the claim that for any relation which is possibly instantiated, says or implies anything about the size of its bearer, and implies that its bearer is composed of points, there are possible intrinsic duplicates that differ

[5] I wish to remain neutral as to whether, if some non-pointy size relations *are* fundamental (and hence intrinsic), they are expressible by one-place or two-placed predicates. See Eddon [2013] for a discussion of that question about quantities more generally.

[6] At least that's what I will mean by 'intrinsic'. Even if Lewis [1983] and others are wrong in claiming that this captures our pre-theoretic notion of intrinsicality – see Eddon [2011] and Marshall [2015] for arguments to that effect – it'll serve my purposes just fine as long the assumptions of the puzzle are plausible so understood.

with respect to it. For example, if Size Extrinsic is true, and some possible two-by-four is composed of points, then there are two possible pieces of lumber that are intrinsic duplicates, one of which, like your usual two-by-four, is 2" × 4" × 8', and one of which isn't.[7] And if my son's 1:32 scale model of a Subaru Impreza and a full size Subaru Impreza are composed of points, then the pair of them has a possible duplicate pair the sizes of whose members don't stand in the ratio 1:32. These consequences are surprising. Why think they're true? More to the point, why think Size Extrinsic is true?

Very roughly, the reason to think it's true is that the standard mathematical theory of measure has it as a consequence. More exactly, the reason to think it's true is as follows. (1) Pointy size is to be understood as – or at least is equivalent to – a physical analogue of the Lebesgue measure (defined on $\mathbb{R}^n$), which I shall call 'Lebesgue$_p$ measure' (Size is Measure).[8] (2) Possibility is combinatorial: no intrinsic nature places any absolutely necessary constraints on its bearer's surroundings (Patchwork Principle).[9] (3) If Patchwork Principle is true then for any possible relation that entails a Lebesgue$_p$ measure, there are two possible intrinsic duplicates that differ with respect to that relation (Measure Extrinsic). So any possible relation that entails a Lebesgue$_p$ measure is extrinsic. So all possible pointy size relations are extrinsic.

This is still a rather bare bones version of the argument, one that is unlikely to win any adherents just as it stands. A fleshed out argument needs to include a detailed justification of Measure Extrinsic. I will present such a justification, but in two stages, or two versions.[10] To understand *why* Measure Extrinsic is true, we need to have a look at the nuts and bolts of measure theory, at how Lebesgue measure, for example, is defined. And I'll present an argument that does just that: it not only justifies but also illuminates the claim that all Lebesgue$_p$ measure relations are extrinsic. But even without an acquaintance with those nuts and bolts, we can see *that* Measure Extrinsic, or something near enough, is true: we can see that if Patchwork Principle is true, then for any possible Lebesgue$_p$

[7] Or, that one is made of points and one isn't. But then they could be intrinsic duplicates only if a point could be an intrinsic duplicate of a non-point (see nt. 24), which would serve just as well as a surprising size-related consequence of Size Extrinsic. In any case, my arguments for Size Extrinsic establish the stronger claims about the examples in the text.

[8] See §2.1.2 for an explicit definition of 'Lebesgue$_p$ measure'. I assume that if pointy size is a physical analogue of a standard mathematical measure, that mathematical measure would be the Lebesgue measure. But I could just as well assume that the relevant mathematical measure would be the Jordan measure, say, and the arguments would be essentially unchanged. (The Lebesgue measure is countably additive, while the Jordan measure is only finitely additive.)

[9] More generally, Patchwork Principle guarantees the possibility of any number of instances of any intrinsic natures in any which arrangement. See Lewis [1986]. For a more careful formulation, see Segal [2015, §2]. (An 'intrinsic nature' is a relation shared by all and only intrinsic duplicates of some possibile. It is equivalent to the conjunction of all its bearer's intrinsic properties.)

[10] I will not provide any justification for Size is Measure or Patchwork Principle; I take the former to be obvious and I've defended the latter elsewhere (Segal [2014]).

measure, there are two possible intrinsic duplicates that differ with respect to it (Measure Extrinsic*). And we can see this simply by attending to the Lebesgue measure of a certain subset of $\mathbb{R}$ and the Lebesgue$_p$ measure of the corresponding subset of a one-dimensional pointy physical space. This provides us with a quick and unenlightening argument for the extrinsicness of pointy sizes. We'll then turn to the slower and more illuminating argument for the fully general conclusion that all pointy size relations are extrinsic.

2.1.1 Quick and Unenlightening Argument

First, for the subset of $\mathbb{R}$. The Cantor Ternary Set is obtained from the interval [0,1] by deleting the open middle third interval, leaving over $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, then deleting the open middle third of each of those two intervals, leaving over $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so on ad infinitum.[11] It contains uncountably many points, but has Lebesge measure 0. Interesting as the Cantor Ternary Set is, it's not exactly what I'm interested in. Rather, I'm interested in a variant that I shall call the 'Right-Open Cantor Ternary Set'. The Right-Open Cantor Ternary Set is obtained from the half-open half-closed interval [0,1) by deleting the half-open, half-closed middle third interval, leaving over $[0, \frac{1}{3}) \cup [\frac{2}{3}, 1)$, then deleting the half-open, half-closed middle third of each of those two intervals, leaving over $[0, \frac{1}{9}) \cup [\frac{2}{9}, \frac{1}{3}) \cup [\frac{2}{3}, \frac{7}{9}) \cup [\frac{8}{9}, 1)$, and so on ad infinitum.[12] The remaining set, like the Cantor Ternary Set, contains uncountably many points, and has Lebesgue measure 0.[13] But in addition, the Right-Open Cantor Ternary Set is topologically isomorphic to any half-open interval, such as the interval [0,1). Both have the (intrinsic) topology of the continuum: they are dense and Dedekind complete.[14]

Next, for the subset of a one-dimensional pointy physical space. Say E is a pointy physical space that is topologically isomorphic to $\mathbb{R}$. Let 'GEORG' name the subset of E that corresponds to the Right-Open Cantor Ternary Set.[15] GEORG contains uncountably many points, and has Lebesgue$_p$ measure 0 meters.[16] But

[11]More formally, let $I_n = \bigcup_{a_i \in \{0,2\}} [\sum_{i=1}^{n} \frac{a_i}{3^i}, \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n}]$. Then the Cantor Ternary Set = $\bigcap_{n=1}^{\infty} I_n$.

[12]More formally, let $I_n = \bigcup_{a_i \in \{0,2\}} [\sum_{i=1}^{n} \frac{a_i}{3^i}, \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n})$. Then the Right-Open Cantor Ternary Set = $\bigcap_{n=1}^{\infty} I_n$.

[13]For the argument, see Appendix A.

[14]For the argument, see Appendix B.

[15]The numerical representation of a point in E is furnished by the length of an interval one of whose endpoints is an arbitrary point in E, the 'origin,' and whose other endpoint is the point in question. This allows us to speak of 'the subset of E that corresponds to ϕ', where 'ϕ' is replaced by the name of a subset of $\mathbb{R}$.

[16]That is, assuming it's possible for something to *have* a Lebesgue$_p$ measure. There is no harm in my assuming that: if no Lebesgue$_p$ measure is possibly instantiated – and so, per Size is Measure, neither is any pointy size – then Size is Extrinsic is vacuously true. Since the assumption is

in addition, GEORG is topologically isomorphic to any half-open interval, such as MATT, a left-closed right-open meter-long interval in *E*. Both have the (intrinsic) topology of the continuum: they are dense and Dedekind complete.

GEORG and its intrinsic duplicates serve as witnesses to the embedded existential claim in the consequent of Measure Extrinsic*. If Patchwork is true, then there is a possible scenario in which an intrinsic duplicate of GEORG *has* no other points between its own. Such an intrinsic duplicate is a non-degenerate interval, as it has everything it takes intrinsically to be a non-degenerate interval, and there isn't anything between its points to get in the way. But note: necessarily, no non-degenerate interval has Lebesgue$_p$ measure 0 meters. That would make hash of the notion of Lebesgue$_p$ measure; it couldn't sensibly be a *measure*. (And since pointy size is equivalent to Lebesgue$_p$ measure, pointy size couldn't sensibly be a *size*.) So that intrinsic duplicate of GEORG does not have Lebesgue$_p$ measure 0 meters. This implies that the relation, **having Lebesgue$_p$ measure 0 meters**, is extrinsic.

Indeed, of necessity every interval in pointy physical space has *positive* Lebesgue$_p$ measure. So GEORG's intrinsic duplicate has positive Lebesgue$_p$ measure. This implies that some positive Lebesgue$_p$ measure is extrinsic. But presumably, what's true for one of GEORG's interval duplicates is true for all of them. There isn't just one positive Lebesgue$_p$ measure that is extrinsic: if it's true for one, it's true for all. Thus, Measure Extrinsic*. And that, together with Size is Measure and Patchwork, implies that all possible pointy sizes are extrinsic.

One might reply by denying that GEORG's intrinsic duplicate is an interval even in a possible scenario in which there are no other points between its own. Here's the only way I can see how one would do so: suppose distance between concrete points in a physical space ('physical distance' for short) is intrinsic. If physical distance is intrinsic, then no possible intrinsic duplicate of GEORG is *congruent* to any possible interval, since GEORGE itself is not congruent to any possible interval.[17] So no possible intrinsic duplicate of GEORG *is* an interval, even one in an environment in which there were no points between its own.

The trouble with this reply is that there is very good reason to think that the lack of congruence between GEORG and any interval *is itself due to what's interspersed between their points*; good reason, more broadly, to think that physical distance is extrinsic. For one thing, according to the standard account of physical distance, what Bricker [1993] calls the "Gaussian conception," it is extrinsic. Here is Maudlin [2014]:

> In a geometrical space, the distance between two points is usually defined as the *the length of the shortest continuous curve that connects them.* (55)

harmless, I will make it at various points in the next paragraph in the text.

[17]Two sets of points are congruent if there is a one-to-one mapping from one to the other that preserves distances. Crucially, congruence is an equivalence relation.

> ...distances are not geometrical primitives: the distance between two points is a derivative notion. More basic than distance is the length of a line connecting the two points: the distance may then be defined as an extremal length (often the shortest length) of all lines that have the points as endpoints. (205)

Clearly enough, if to say that the distance between *a* and *b* is ϕ is to say something about the length of all the curves that connect *a* and *b*, then distance relations are extrinsic. As Bricker [1993] elaborates,

> On the Gaussian conception...the distance between two points turns out not to be an intrinsic relation of those points. If some of the space surrounding two points is "removed," some or all of the paths connecting those points may no longer exist, and the length of shortest remaining path – the new distance between the points – may be greater than it was, or not defined. If the space surrounding two points is embedded in a larger space, new paths connecting the points may come into existence, and the length of the shortest connecting path – the new distance between the points – may be less than it was. In short: the distance between two points does not depend solely upon the intrinsic nature of the fusion of the two points. (27)[18]

For another thing, there are good reasons that derive from Patchwork to think physical distance is extrinsic. For instance, the triangle inequality, widely held to be partly constitutive of the notion of distance, imposes an absolutely necessary constraint on the distance between two points, given the distance between each of those points and a third point. If physical distance is intrinsic, this necessary constraint violates Patchwork.[19]

The upshot: it's very hard to salvage the intrinsicness of size by resorting to the intrinsicness of distance, since the latter idea is both non-standard and Humeanly intolerable. And if you are prepared to accept what a Humean cannot tolerate, then you might as well object to the argument on those grounds alone; no need to drag distance into it.

The Quick and Unenlightening argument strongly suggests that pointy size is extrinsic. However, being quick and unenlightening, it doesn't explain *why* that is,

[18]Note: if the standard Gaussian account is correct, then contrary to the impression one might get from Bricker's remarks distance relations are extrinsic even to pairs of points in a one-dimensional space (where there is only one path that connects them), since the length of the path that connects them is a matter of how much lies *between* the two points and whether the two points lie in a one-dimensional space or a multi-dimensional space is a matter that is itself extrinsic to the points.

[19]See Maudlin [2007, 88-9]. Maudlin and I are assuming a version of Patchwork that guarantees possibilities that involve overlap (at least under certain conditions, conditions which are met in this case); see Segal [2015, §2] for details.

and it doesn't establish that *all* pointy size relations are extrinsic. For illumination and generality we turn to the Slower and More Illuminating Argument, which begins with a definition of Lebesgue measure.

2.1.2 Slower and More Illuminating Argument

How is the Lebesgue measure defined? What does it mean to say, for example, that the Lebesgue measure $m(S)$ of a certain subset S of $\mathbb{R}$ equals 2? (For the sake of simplicity, I restrict my attention throughout the remainder of my discussion to one-dimensional spaces and the size relations their regions instantiate; for one thing, the topology of a one-dimensional space is fixed by an ordering of its points. But the argument can easily be generalized.) We start with an assignment of a real number to each of the intervals. For any interval I in $\mathbb{R}$ from a to b, its so-called elementary measure, $m(I)$, is simply $b - a$. In order to expand our assignment of measure beyond just intervals, we first introduce the notion of a Lebesgue *outer* measure m^*, which applies to any arbitrary subset S of $\mathbb{R}$: we say that $m^*(S)$ is the minimal, or more exactly infimal, cost required to cover S by a countable union of intervals. In other symbols,

$$m^*(S) = \inf_{S \subseteq \bigcup_{i=1}^{\infty} I_i; I_i interval} \sum_{i=1}^{\infty} m(I_i)$$

Finally, for any Lebesgue measurable set S the Lebesgue measure of S just *is* the Lebesgue outer measure of S.

What's critical to notice for our purposes is that the Lebesgue outer measure of at least some subsets of $\mathbb{R}$ is at least partly a matter of the size of sets that *properly include* them. Hence the apt name, 'Lebesgue *outer* measure'. In particular, the outer measure of any subset of $\mathbb{R}$ that isn't itself a countable union of intervals is partly determined by the nature of its surroundings.

And the same goes for subsets of a one-dimensional pointy physical space, like E. We can define Lebesgue$_p$ measure in just the same way as Lebesgue measure. We start with an assignment of a real number to each of the intervals. For any interval I in E from a to b, its so-called elementary$_p$ measure, $m_p(I)$, is simply its length-in-meters (or in-cubits, or in-some-unit-or-other).[20] The Lebesgue$_p$ outer

[20]That is, assuming it's possible for something pointy to have a length. But as I remarked in nt. 16, there is no harm in my assuming that: if no pointy size is possible, then Size is Extrinsic is vacuously true. Since the assumption is harmless, I will make it at various points in the next paragraph in the text.

N.B. There *is* something puzzling here. Since we have argued that physical distance is extrinsic, you might reasonably deduce from my discussion that elementary$_p$ measure, the length of a pointy interval, is fundamental (and hence intrinsic). But it can't be, as we'll see: Patchwork won't allow it. This is puzzling: where *does* a pointy interval's elementary$_p$ measure, its length, come from? But noting this does nothing to impugn the coherence of Lebesgue$_p$ measure and with it, the force measure m_p*(S) of any arbitrary subset S of E is the minimal, or more exactly infimal, cost required to cover S by a countable union of intervals. In other symbols,

$$m_p^*(S) = \inf_{S \subseteq \bigcup_{i=1}^{\infty} I_i; I_i interval} \sum_{i=1}^{\infty} m_p(I_i)$$

Finally, for any Lebesgue$_p$ measurable set S the Lebesgue$_p$ measure of S just *is* the Lebesgue$_p$ outer measure of S. And just as with regard to the Lebesgue measure, the Lebesgue$_p$ outer measure of some subsets of E is at least partly a matter of the size of sets that *properly include* them. In particular, the outer measure of any subset of E that isn't itself a countable union of intervals is partly determined by the nature of its surroundings. Modify those surroundings sufficiently, but hold fixed a thing's intrinsic nature and its Lebesgue$_p$ outer measure will change accordingly.

Lo and behold, GEORG and its intrinsic duplicate provide a good illustration (although by no means the only). GEORG is not a countable union of intervals.[21] Its Lebesgue$_p$ outer measure of 0 meters is therefore partly determined by the nature of its surroundings. Modify those surroundings sufficiently, say by removing all the other points between its own, but hold fixed its intrinsic nature, and the Lebesgue$_p$ outer measure of that set will not be 0 meters. And if Patchwork is true, then such a modification is guaranteed to be possible: there *is* a possible scenario in which an intrinsic duplicate of GEORG has no other points between its own. Such an intrinsic duplicate is a non-degenerate interval, and so has positive Lebesgue$_p$ outer measure. So Lebesgue$_p$ outer measure is extrinsic. More importantly, since GEORG is Lebesgue$_p$ measurable (as a bonus, so is the intrinsic duplicate we're considering), Lebesgue$_p$ measure itself is extrinsic.

This already allows us to give an argument for the conclusion that all possible pointy sizes are extrinsic – much like the Quick and Unenlightening Argument – and also explains *why* they are extrinsic. But what of possible relations that entail them? Given the foregoing argument, one might happily concede that pointy sizes are extrinsic but object that that fact provides no support for the more general claim. Indeed, the details of the argument readily suggest a class of possible pointy size relations that are intrinsic. I have repeatedly restricted our attention to those subsets of E that can't be exactly covered by a countable collection of intervals. When we instead concentrate on those that can, it would appear that their Lebesgue$_p$ outer measure, and hence their Lebesgue$_p$ measure, is an entirely

of our puzzle; it simply underscores another aspect of it. As I said, if intervals in a one-dimensional pointy physical space *have* lengths, then given those lengths, from wherever they come, Lebesgue$_p$ measure can be straightforwardly defined in just the same way as Lebesgue measure to generate our puzzle.

[21] As we've noted, it contains uncountably many points. So it is a countable union of intervals only if it includes at least one non-degenerate interval. But if it includes at least one non-degenerate interval then it does not have Lebesgue$_p$ measure 0 meters, as it does.

intrinsic matter. What difference could the surroundings of a set S make to its Lebesgue$_p$ outer measure if S just *is* a countable union of intervals? Thus, the idea goes, if, and only if, S is a countable union of intervals, any possible intrinsic duplicate of S has the same Lebesgue$_p$ measure as S. From which it follows that any relation expressible by an instance of the following schema is intrinsic[22]:

> **being an x such that (a) x is a countable union of intervals and (b) x has a Lebesgue$_p$ measure of ϕ meters**

If this is right, then Size Extrinsic is false, since every instance of that schema is equivalent to a possible pointy size relation.[23] And if it's right, no deep puzzle would arise, since the distribution of *those* pointy size relations, along with the fundamental topological, affine, etc. relations, suffices to settle the distribution of pointy sizes.

But it isn't right. The objection ignores the fact that whether a set is a countable union of intervals is itself an extrinsic matter. GEORG, for example (yet again!), is not a countable union of intervals. But as we've seen, if Patchwork is true, then GEORG has an intrinsic duplicate that is not only a countable union of intervals, but an interval. So there are intrinsic duplicates that differ with respect to some instance of the relation schema,

> **being an x such that (a) x is a countable union of intervals and (b) x has a Lebesgue$_p$ measure of ϕ meters**,

pace the objection. And the very same point can be made in response to the suggestion that instances of the following relation schema (i.e. elementary$_p$ measures) are intrinsic:

> **being an x such that (a) x is an interval and (b) x has a Lebesgue$_p$ measure of ϕ meters**

Since there don't seem to be any other decent candidates for possible intrinsic relations that entail a Lebesgue$_p$ measure, Measure Extrinsic itself – and not just

[22] Suppose something S has a relation expressible by an instance of the schema: for the sake of definiteness, let '1' substitute for 'ϕ'. Then S is a countable union of intervals, so, by assumption, every possible intrinsic duplicate of S has the same Lebesgue$_p$ measure as S; that is, every possible intrinsic duplicate has a Lebesgue$_p$ measure of one meter . Furthermore, every possible intrinsic duplicate of S is a countable union of intervals. For suppose there is some possible intrinsic duplicate S' of S – and so of Lebesgue$_p$ measure of one meter – that is not a countable union of intervals. Then, by the 'only if' half of the assumption, S' has an intrinsic duplicate S'' that does not have a Lebesgue$_p$ measure of one meter. But intrinsic duplication is transitive, so S'' is an intrinsic duplicate of S, which contradicts the assumption that every possible intrinsic duplicate of S has a Lebesgue$_p$ measure of one meter. Thus, every possible intrinsic duplicate of something that has the relation, **being an x such that (a) x is a countable union of intervals and (b) x has a Lebesgue$_p$ measure one meter** also has that relation. And so the relation is intrinsic.

[23] At least if pointy size relations are in fact possible. And we will have been given no reason to think they aren't.

Measure Extrinsic* – seems very difficult to deny.[24] And as we've noted, that, together with Size is Measure and Patchwork, implies that all possible pointy size relations are extrinsic.

Of course, neither the Quick and Unenlightening Argument nor the Slower and More Illuminating Argument show that *non*-pointy size relations are extrinsic.[25] More generally, we have no reason anywhere in the vicinity of these arguments to think that non-pointy size relations are extrinsic. Pointy space has two rivals: gunky space and chunky space. In the former sort of space, there is no minimal size, and in the latter sort, there is a minimal positive size.[26] In either case, there are no regions of size 0, and hence no region of finite size can be decomposed into more than countably many parts. Since size is countably additive (or at least it's ordinarily taken to be), we should accept the principle, which we couldn't accept for pointy space, that for anything x and any collection of things that compose x, the size of x is the sum of the sizes of the members of that collection.[27] And given that principle, there is no longer any reason along the lines I have sketched to believe that the size of such regions or objects is extrinsic. The size of any such region or object is always strictly determined by the sizes of its parts, and the size of each of those parts is in turn strictly determined by the sizes of *its* parts, and so on: it won't matter what else you throw in between them.[28]

2.2 Assumption 2

The second assumption is as follows:

> (Extrinsic Supervenes) The set of extrinsic relations globally supervenes on the set of intrinsic relations

[24]What about possible pointy size relations that entail **being a point** or **being a countable collection of points**, and hence **having Lebesgue$_\mathbf{p}$ measure 0**? Might those be intrinsic and possible?

Perhaps they would be. But the larger puzzle would be unaffected by the concession that only those possible pointy size relations are intrinsic, for the distribution of those relations, together with that of the fundamental topological, affine, etc. relations won't suffice to settle the distribution of the rest of the pointy size relations. And aside from those, there don't seem to be any even half-decent candidates for intrinsic possible pointy size relations.

[25]A 'non-pointy size relation' is a size relation that entails that none of its relata is composed of points.

[26]I have in mind the measure-theoretic sense of both 'gunky' and 'chunky'. Each conception has an illustrious history and has recently received some support, or at least some positive press, or at least some consideration as an interesting possibility worthy of investigation. For discussion of gunk, see Skyrms [1993], Zimmerman [1996], Hawthorne and Weatherson [2004], Arntzenius and Hawthorne [2005], Arntzenius [2008], and Russell [2008]. For some discussion of chunk, see Simons [2004], Forrest [1995], and Forrest [2004].

[27]See Arntzenius and Hawthorne [2005], where this principle is called 'Summing'. For discussion of whether size is indeed countably additive, see Russell [2008].

[28]For a fully developed mathematical theory of measure for pointless spaces, see Caratheodory [2011]. The measure Cartheodory defines is intrinsic.

What I mean by 'global supervenience' is as follows: where A is a set of relations, let an *A-isomorphism* be a one-to-one mapping that is isomorphic with respect to every relation in A. Then,

> 'set A globally supervenes on set B' $=_{\text{df}}$ for any worlds w_1 and w_2, every B-isomorphism from w_1's domain onto w_2's domain is an A-isomorphism.[29]

So, Extrinsic Supervenes amounts to the claim that the distribution of extrinsic relations can't float free – it's always settled by the distribution *over everything there is* of the intrinsic relations.

The argument for this claim is extremely straightforward. If two possible Worlds, W_1 and W_2 are intrinsic duplicates, then the two Worlds do not differ at all, except perhaps in a non-qualitative way.[30] After all, they are intrinsic duplicates and neither is accompanied by anything disjoint from itself. But then Extrinsic Supervenes follows immediately.[31]

2.3 Assumption 3

The third and final assumption is this:

> (Size Floats Free) If there is a possible pointy size relation, then the set of possible pointy size relations does not globally supervene on the set of intrinsic non-pointy-size relations[32]

This assumption should, at this point, hardly require any defense. Since I have already argued that distance relations are not intrinsic, it seems to be the case that the set of possible pointy size relations globally supervenes on the set of intrinsic non-pointy-size relations only if either there *are* no possible pointy size relations or the sub-metrical structure of a pointy physical space is all the structure to be had. And the latter disjunct is obviously false, on both *a priori* and *a posteriori* grounds. *A priori* grounds: one can rather easily conceive of two pointy physical spaces such that a sub-metrical isomorphism from the domain of one to the

[29]This is equivalent to what Sider [1999] calls 'strong global supervenience'; see his nt. 10.

[30]I am using 'World' (capitalized) to mean '*concrete* world,' i.e. 'whole Cosmos', not an ersatz or linguistic representation of such. And to avoid taking any controversial mereological positions, I mean to use that latter phrase to denote a sequence of all the individuals there are, rather than a mereological sum of them.

[31]Let 'B' name the set of intrinsic relations, and 'A' name the set of all (qualitative) relations. Suppose f is a B-isomorphism from the domain of w_1 to w_2. Then there is some sequence W_1 of all the individuals in w_1 and some sequence W_2 of all the individuals in w_2, such that W_2 is the image (under f) of W_1, and (consequently) W_1 and W_2 are intrinsic duplicates. But since W_1 and W_2 are also Worlds, they share all qualitative relations (by the assumption in the text). So then f is an A-isomorphism.

[32]A 'non-pointy-size relation' (not to be confused with a non-pointy size relation – see nt. 25) is one that is not a pointy size relation.

domain of the other isn't a measure-theoretic isomorphism. *A posteriori* grounds: no empirically adequate theory of physical reality has ever been proposed that relies only on the sub-metrical structure of space.[33] So if there *are* possible pointy size relations, they don't globally supervene on the set of intrinsic non-pointy-size relations.

2.4 Pulling the Puzzle Together

As should hopefully be clear, our three assumptions – Size Extrinsic, Extrinsic Supervenes, and Size Floats Free – are inconsistent with there being any possible pointy size relations. Given Size Extrinsic and Extrinsic Supervenes, the set of possible pointy size relations globally supervenes on the set of intrinsic relations;[34] but again, given Size Extrinsic, the set of intrinsic relations is the same set as the set of intrinsic *non*-pointy-size relations. So the set of possible pointy size relations globally supervenes on the set of intrinsic non-pointy-size relations. But, if there is a possible pointy size relation, then according to Size Floats Free, they don't supervene on the set of intrinsic non-pointy-size relations. So there is no possible pointy size relation.

The puzzle is now in place. It suggests that no pointy size relation could be instantiated, that it's not possible for something that is made of points to have any size at all. Before I consider a resolution, let me briefly discuss the applicability of our puzzle to mathematical spaces. One might think that if there is a puzzle about physical spaces of concrete points, then *a fortiori* there is a puzzle about mathematical spaces of abstract points, since the measure functions I have been discussing were defined explicitly regarding such spaces. If that's true, then so be it. But I am not at all convinced that it is true. Several replies can be given in the case of the mathematical spaces but not in the case of physical spaces. Here I mention two:

1. Nominalism – there are no abstracta. And if there are no abstracta, then there are no mathematical spaces made of abstract points. And if there are no such spaces, then of course there are no regions to instantiate measure-theoretic relations, either extrinsically or intrinsically.

2. Restricted Patchwork – there are abstracta, but Patchwork therefore needs to be restricted to concrete reality. This makes room for several places to balk at my arguments, were they directed at mathematical spaces. For example, we might say that distance between abstract points in a mathematical

[33]Since these grounds are *a posteriori*, they establish only a conditional: if our space is in fact pointy, then Size Floats Free is true.

[34]If there are no possible pointy size relations, the set of pointy size relations is empty, and supervenes on any set whatsoever.

space ('mathematical distance' for short) is intrinsic, and perhaps even internal (in the sense that it is settled by the intrinsic natures of the relata).[35] Since mathematical distance is intrinsic, a version of Size Extrinsic about the measure of mathematical regions is false, as is a version of Size Floats Free about mathematical spaces. The fact that mathematical distance is intrinsic doesn't run afoul of the Patchwork Principle, despite mathematical distance's obedience to the triangle inequality and other related constraints, since the Patchwork Principle is restricted to concrete reality.

Now back to our original puzzle and a possible resolution.

3 Possible Resolution: Size Doesn't Float Free

As I said in my brief defense of Size Floats Free, since I have already argued that distance relations are extrinsic, it seems to be the case that Size Floats Free is false only if the sub-metrical structure of a pointy physical space is all the structure to be had. But maybe things aren't how they seem. How could pointy size not float free and yet be an independent source of structure?

I see only one way. Perhaps there are intrinsic relations whose distribution settles the pointy size facts, but they are not instantiated by regions of ordinary, pointy space (or spacetime). They are instantiated by something else: an underlying space, a Mind ... *something or other*, whose existence and nature determines the existence and nature of our ordinary, pointy space (or spacetime). The intrinsic character of this other thing determines the pointy size facts, so Size Floats Free is false; but that in no way implies that the sub-metrical structure is all the structure to be had (even in pointy spaces). Two possible pointy spaces can differ in their metrical structure without differing in their sub-metrical structure: they will just differ with respect to the intrinsic character of something else that accompanies them.

This resolution has at least this going for it: its chief implication for fundamental reality – viz., that no pointy space has any part in it – agrees with that of leading theories in quantum gravity.[36] But it goes further than such theories in denying the *possibility* of a fundamental pointy space, or at least of one whose regions have sizes. Mathematical physicists had assumed for some time and as a matter of course that physical space is fundamental and pointy. If we have only this way out of the puzzle, then their assumption couldn't so much as possibly have been true. A pointy physical space, whose regions have sizes, can be no part

[35]See Maudlin [2014, pp. 6-9] for the claim that each point in $\mathbb{R}^n$ has a rich intrinsic nature, an intrinsic nature that entails such-and-such a metrical relation to points with thus-and-such intrinsic nature.

[36]See, inter alia, Huggett and Wuthrich [2013].

of fundamental reality. If that's not a devastating problem for points, it's a rather significant blow.[37]

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A Measure 0 and Uncountable

In this Appendix I prove that the Right-Open Cantor Ternary Set has Lebesgue measure 0 and contains uncountably many points.
(a) Lebesgue measure 0: the Right-Open Cantor Ternary Set is a proper subset of the Cantor Ternary Set. Lebesgue outer measure is monotonic – so the Lebesgue outer measure of the Right-Open Cantor Ternary Set is ≤ 0, and so it is equal to 0. Every set of Lebesgue outer measure 0 is measurable [Tao, 2011, p. 26], so its Lebesgue measure is itself 0.
(b) Uncountably many points: the Cantor Ternary Set is the union of the Right-Open Cantor Ternary Set and the Left-Open Cantor Ternary Set (defined in the natural way); by considerations of symmetry, the Right-Open Cantor Ternary Set is countable iff the Left-Open Ternary Set is countable. But the union of two countable sets is countable, so if the Right-Open Cantor Ternary Set is countable, so is the Cantor Ternary Set, contrary to fact.

B Dedekind Complete and Dense

In this Appendix I prove that the Right-Open Cantor Ternary Set is both Dedekind complete and dense.
(a) Dedekind completeness: suppose there is a partition of the Right-Open Cantor Ternary Set into two sets, A_1 and A_2, such that every member of A_1 is less than every member of A_2, but there is no member of the Right-Open Cantor Ternary Set that is both greater than or equal to every member of A_1 and less than or equal to every member of A_2. (That is, suppose it's not Dedekind complete.) But since the real-number interval (0,1) *is* Dedekind complete, there is *some* real number r, $0 < r < 1$, such that r is both greater than or equal to every member of A_1 and less than or equal to every member of A_2. So r is not a member of the Right-Open Cantor Ternary Set. It must have been "removed" at some point, i.e. it must belong to some half-open interval that contains no members of the Right-Open Cantor Ternary Set but whose right endpoint *is* a member of the Right-Open Cantor Ternary Set. But then the right endpoint of that half open interval is a member of the Right-Open Cantor Ternary Set that is greater than or equal to every member of A_1 and less than or equal to every member of A_2, contrary to our assumption. Reductio.
(b) Density: for any two members of the Right-Open Cantor Ternary Set, x and y, $x < y$, there is an interval $[c, d)$ – where $x < c$, $d \leq y$ – such that for some n, $c = \sum_{i=1; a_i \in \{0,2\}}^{n} \frac{a_i}{3^i} + \frac{1}{3^n}$. (The interval [c,d) is one of the intervals that was removed. There must be some such interval since there is some point in the complement that is between x and y – otherwise the Right-Open Cantor Ternary Set would include

a non-degenerate interval, which it doesn't – and at every stage at which a point was removed, a left-closed right-open interval was removed, whose left endpoint is equal to some such sum.) But for some m, c - x > $\frac{1}{3^m}$. If m < n, then c - x > $\frac{1}{3^n}$; so x < (c - $\frac{1}{3^n}$) < c; and c - $\frac{1}{3^n}$ = $\sum\limits_{i=1; a_i \in \{0,2\}}^{n} \frac{a_i}{3^i}$ $\in$ Right-Open Cantor Ternary Set, so there is a member of the Right-Open Ternary Set between x and y. If $m \geq n$, then c = $\sum\limits_{i=1; a_i \in \{0,2\}}^{m} \frac{a_i}{3^i} + \frac{1}{3^m}$ (for any i > n, let a_i = 2; keep all other a_i the same). So x < c - $\frac{1}{3^m}$ < c; and c - $\frac{1}{3^m}$ = $\sum\limits_{i=1; a_i \in \{0,2\}}^{m} \frac{a_i}{3^i}$ $\in$ Right-Open Cantor Ternary Set, so there is a member of the Right-Open Ternary Set between x and y. Either way, there is some member of the Right-Open Cantor Ternary Set that is between x and y.